Overview of the theory of quadratic forms

All the fields will be of characteristic different from 2.

1. Basic notions

1.1. **Definitions.** A *quadratic form* on a k-vector space V is a map $q : V \to k$ such that

- $q(\alpha v) = \alpha^2 q(V)$ for all $\alpha \in k$ and $v \in V$;
- the map $V \times V \to k$, $(v, w) \mapsto q(v + w) - q(v) - q(w)$, is bilinear.

A *quadratic space* is a pair (V, q) given by a quadratic form on a vector space V, but we usually identify (V, q) with the form q. The space $V = V_q$ is called the *underlying space* of q and $\dim V$ the *dimension* of q, denoted by $\dim q$.

Given a quadratic form q, the map $b_q : \begin{cases} V \times V & \to & k \\ (v, w) & \mapsto & \frac{1}{2}(q(v + w) - q(v) - q(w)) \end{cases}$ is a symmetric bilinear form called the *polar form* of q. We have $b_q(v, v) = q(v)$ for all $v \in V_q$.

The quadratic form q is said to be *non degenerated* if for all $v \in V_q$, the fact that $b_q(v, w) = 0$ for all $w \in V_q$ implies $v = 0$. The underlying space V_q being finite dimensional, the non degeneracy of q is equivalent to the fact that the linear map

$$\begin{array}{ccc} V_q & \to & V_q^* \\ v & \mapsto & (w \mapsto b_q(v, w)) \end{array}$$

is an isomorphism.

Example 1.1. Let $a_1, \ldots, a_n \in k$. We define $\langle a_1, \ldots, a_n \rangle$ to be the quadratic space given by the vector space k^n endowed with $k^n \to k$, $(x_1, \ldots, x_n) \to a_1 x_1^2 + \ldots + a_n x_n^2$. Such a quadratic space is non degenerate if, and only if, all a_i are non zero.

An *isometry* $(V, q) \to (V', q')$ is a bijective linear map $f : V \to V'$ such that $q'(f(v)) = q(v)$ for all $v \in V$. If (V, q) and (V', q') are isometric, we write $(V, q) \simeq (V', q')$.

For the remaining of the paper, we shall only consider non degenerate quadratic spaces of finite dimension.

Proposition 1.2. *Every quadratic space is isometric to some* $\langle a_1, \dots, a_n\rangle$.

The *orthogonal group* associated to q is the group $O(q)$ formed by the isometry $(V,q) \to (V,q)$. In $O(q)$ we distinguish special isometries, called *orthogonal reflections* : Let $a \in V_q$ such that $q(a) = b_q(a,a) \neq 0$; then the orthogonal reflection with respect to a is the isometry r_a given by the formula

$$r_a(v) = v - 2\frac{b_q(a,v)}{b_q(a,a)}a, \ \forall v \in V_q.$$

Proposition 1.3. *For any quadratic space q, the group $O(q)$ is generated by orthogonal reflections.*

1.2. **Operations on quadratic forms.**

Sums of quadratic forms : Let $(V_1, q_1), \dots, (V_n, q_n)$ be quadratic spaces. Then we define $\perp_i q_i = \perp_i (V_i, q_i)$ to be the quadratic form q on $V = \bigoplus_i V_i$ defined by the formula

$$q(v_1, \dots, v_n) = q_1(v_1) + \ldots + q_n(v_n), \ \forall v_1 \in V_1, \dots, \forall v_n \in V_n.$$

For instance, given $a_1, \dots, a_m, b_1, \dots, b_n \in k^\times$, we have an isometry

$$\langle a_1, \dots, a_m\rangle \perp \langle b_1, \dots, b_n\rangle \simeq \langle a_1, \dots, a_m, b_1, \dots, b_n\rangle.$$

Proposition 1.4. *Let q, q_1, q_2 be quadratic forms. If $q_1 \perp q \simeq q_2 \perp q$, then $q_1 \simeq q_2$.*

Products of quadratic forms : Let $(V_1, q_1), \dots, (V_n, q_n)$ be quadratic spaces. Then we define $\bigotimes_i q_i = \bigotimes_i (V_i, q_i)$ to be the quadratic form q on $V = \bigotimes V_i$ whose polar form is characterized by the formula

$$b_q(v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n) = b_{q_1}(v_1, w_1) \cdots b_{q_n}(v_n, w_n) \ \forall v_i, w_i \in V_i.$$

For instance, given $a_1, \dots, a_m, b_1, \dots, b_n \in k^\times$, we have an isometry

$$\langle a_1, \dots, a_m\rangle \otimes \langle b_1, \dots, b_n\rangle \simeq \langle a_1 b_1, \dots, a_i b_j, \dots, a_m b_n\rangle.$$

Scalar extensions of quadratic forms : Let (V,q) be a quadratic space and fix a field extension K/k. Then we define $q_K = (V,q)_K$ as the quadratic form over K defined on $V \otimes_k K$ whose polar form is characterized by the formula

$$b_{q_K}(v \otimes \lambda, w \otimes \mu) = \lambda\mu b_q(v,w), \ \forall \lambda, \mu \in K, \forall v, w \in V.$$

An important particular case of scalar extension is the extension by the *function field* of a given quadratic space (V,q). To define it, consider a basis $v_1, \dots, v_n$ of V.

With respect this basis, the expression of q is

$$q(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \sum_{i,j} a_{ij}\lambda_i\lambda_j,\ \forall \lambda_1, \ldots, \lambda_n \in k,$$

where the a_{ij}'s are certain coefficients. If $\dim q \geqslant 3$, we can prove that the homogeneous polynomial $Q = \sum_{ij} a_{ij}X_iX_j$ is irreducible in $k[X_1, \ldots, X_n]$ and we get a field $k(q) = k[X_1, \ldots, X_n]/Q$ which does not depend on the choices of the basis $v_1, \ldots, v_n$ and the coefficients a_{ij}. We can then consider $q_{k(q)}$ which gives information on q.

1.3. **Isotropy and the hyperbolic plane.**

Definitions

A quadratic form q is said to be *isotropic* if there exists $v \in V_q \setminus \{0\}$ such that $q(v) = 0$. If no such v exists, q is said to be *anisotropic*.

Note that isotropy can be related to the elements of $k^\times$ that a given quadratic form q represents. Indeed, an element $a \in k^\times$ is of the form $q(v)$ for some $v \in V_q$ if, and only if, the quadratic form $q \perp \langle -a \rangle$ is isotropic.

The *hyperbolic plane* is the quadratic space $\mathbb{H}$ given by the k-vector space k^2 and the quadratic form $q(x, y) = xy$. It is isotropic.

Proposition 1.5. *For all $a \in k^\times$, $\mathbb{H} \simeq \langle a, -a \rangle$.*

The hyperbolic plane is the core of isotropy :

Theorem 1.6.

- *Every 2 dimensional isotropic quadratic form is isometric to $\mathbb{H}$.*
- *(Witt decomposition) Every quadratic form q has a decomposition*

$$q \simeq \underbrace{\mathbb{H} \perp \ldots \perp \mathbb{H}}_{m\ times} \perp q_{an}$$

where q_{an} is anisotropic. Moreover, the integer m and the isometry class of q_{an} don't depend on such a decomposition.

The form q_{an} is called the *anisotropic part* of q.

Isotropy and scalar extensions

There is a way to make any quadratic form isotropic : For all quadratic forms q, $q_{k(q)}$ is isotropic.

Also, scalar extensions enables to detect when a quadratic form is isotropic on its base field.

$\diamond$ Consider the transcendental extension $k(X)/k$. Then a quadratic form q over k such that $q_{k(X)}$ is isotropic is actually isotropic over k.

$\diamond$ (Springer's Theorem) Let q be a quadratic form over k. If q_K is isotropic for an odd degree extension K/k, then q is isotropic over k.

$\diamond$ (Hasse-Minkowski Theorem) Let q be a quadratic form over the field of rational numbers $\mathbb{Q}$. Then q is isotropic if, and only if, q_K is so for $K = \mathbb{R}$ and $K = \mathbb{Q}_p$ for all prime number p.

2. Witt rings

The Witt ring of the field k is used to study k through the anisotropicity of quadratic forms over k.

Two quadratic forms q and q' are said to be *Witt-equivalent* if $q_{\text{an}} \simeq q'_{\text{an}}$. This is equivalent to the property : there exists integers m and m' such that $q \perp m\mathbb{H} \simeq q' \perp m'\mathbb{H}$.

Define $W(k)$ to be the set of Witt-equivalence classes of quadratic forms over k. For a given quadratic form q, denote its Witt-equivalence class by $[q]$. This is also the set of isometry classes of anisotropic forms.

We endow $W(k)$ with an addition and a multiplication that make $W(k)$ a ring. This yields the *Witt ring* of k.

Addition : Define the sum $[q] + [q'] = [q + q']$.

Multiplication : Define the product $[q] \cdot [q'] = [q \otimes q']$.

The sum makes $W(k)$ a group. The zero element is $[\mathbb{H}]$, and the inverse of $[q]$ is $-[q] = [-q]$ where $-q$ is the space V_q endowed with the map $v \in V_q \mapsto -q(v)$.

The product turns $(W(k), +)$ into a ring whose unit element is $[\langle 1\rangle]$.

Example 2.1.

- $W(\mathbb{R}) \simeq \mathbb{Z}$ generated by $[\langle 1\rangle]$;
- When $k^2 = k$, $W(k) \simeq \mathbb{Z}/2$;
- For finite fields $\mathbb{F}_q$, $W(\mathbb{F}_q) \simeq \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}/2, & \text{if } q \equiv 1 \bmod 4 \\ \mathbb{Z}/4, & \text{if } q \equiv 3 \bmod 4 \end{cases}$;
- For formally real fields k (that is -1 is not a sum of squares), if $|k^\times/{k^\times}^2| = 2$, then $W(k) \simeq \mathbb{Z}$; in general $W(k)$ has no torsion;
- We have $W(\mathbb{Q}) = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \bigoplus_{p\neq 2 \text{ prime}} W(\mathbb{F}_p)$.

Given a field extension K/k, extension of scalar induces a ring homomorphism

$$\begin{cases} W(k) & \to & W(K) \\ [q] & \mapsto & [q_K] \end{cases}.$$

Example 2.2.

- If K/k is purely transcendental or finite of odd degree, then $W(k) \to W(K)$ is injective.
- We have $W(\mathbb{Q}) \hookrightarrow W(\mathbb{R}) \oplus \bigoplus_{p \text{ prime}} W(\mathbb{Q}_p)$.

For the following result, define for any quadratic form q, the set

$$D(q) = \{q(v)k^{\times 2} \mid v \in V_q,\ q(v) \neq 0\} \subset k^\times/k^{\times 2}.$$

Theorem 2.3 (Harrison-Cordes theorem)**.** *Two fields k_1, k_2 have isomorphic Witt rings if, and only if, there exists a group isomorphism $t : k_1^\times/k_1^{\times 2} \to k_2^\times/k_2^{\times 2}$ such that*

- $t(-1) = -1$;
- *for all $a \in k^\times$, $t(D(\langle 1, a\rangle)) = D(\langle 1, t(a)\rangle)$ inside $k_2^\times/k_2^{\times 2}$.*

Fundamental ideal of the Witt ring. For any two forms q, $q' \in x$ which are Witt equivalent, there exist integers m, m' for which $q \perp m\mathbb{H} \simeq q' \perp m'\mathbb{H}$. Since $\mathbb{H}$ has dimension 2, $\dim q$ and $\dim q'$ have the same parity. Hence we can define

$$\rho : \left\{ \begin{array}{ccc} W(k) & \to & \mathbb{Z}/2 \\ [q] & \mapsto & \dim q \end{array} \right.$$

which is a ring homomorphism. The kernel $I(k)$ of ρ is called the *fundamental ideal* of $W(k)$. We also define the ideal $I^n(k)$ given by the product of n times the fundamental ideal.

An n-Pfister form is a quadratic form isometry to a product

$$\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle.$$

The Proposition says $I(k)$ is additively generated by the 1-Pfister forms.

Proposition 2.4. *The additive group $I^n(k)$ is generated by the n-Pfister forms.*

Theorem 2.5 (Arason-Pfister's Hauptsatz)**.** *Every anisotropic quadratic form whose Witt class lies in $I^n(k)$ has dimension $\geqslant 2^n$. In particular, $\cap_{n\geqslant 0} I^n(k) = 0$.*

3. Field invariants

3.1. **Level of a field.** Define $s(k)$, the *level* of k, as the minimal integer n such that -1 is a sum of n squares in k.

Example 3.1.

- k is formally real if, and only if, $s(k) = +\infty$.
- If k is quadracally closedm then $s(k) = 1$.
- If k is a finite field $\mathbb{F}_q$, then $s(k) = \left\{ \begin{array}{ll} 1 & \text{if } q \equiv 1 \bmod 4 \\ 2 & \text{if } q \equiv 3 \bmod 4 \end{array} \right.$. It follows that $s(k) > 0$ implies $\mathrm{char} k = 0$.

Given a field extension K/k, of course $s(k) \geqslant s(K)$. When K is purely transcendental, $s(K) = s(k)$.

Theorem 3.2. *If k is not formally real, then $s(k)$ is a power of 2.*

Proposition 3.3. *If $s(k) = 2^n$, then the Witt group $W(k)$ is 2^{n+1} torsion.*

3.2. **u-invariant.** The u-invariant of k is $u(k)$, the highest integer n such that there exists an n dimensional anisotropic quadratic form over k.

Example 3.4.

- $u(\mathbb{R}) = u(\mathbb{Q}) = +\infty$ (it suffices to consider the n dimensional form $\langle 1, \ldots, 1\rangle$).
- Of course, $u(\mathbb{C}) = 1$.
- For finite fields, $u(\mathbb{F}_q) = 2$ (Chevalley–Warning theorem).
- The field $\mathbb{C}(X_1, \ldots, X_n)$ of fractions with n indeterminates has u-invariant equal to 2^n.

Proposition 3.5. *If k is not formally real, then $u(k)$ cannot be 3, 5, 7.*

Question : Is $u(k)$ always a power of 2 ?
Answer : No.

Theorem 3.6 (Merkurjev)**.** *Every even integer is the u-invariant of a field.*

Theorem 3.7 (Vishik)**.** *For all $n \geqslant 3$, $2^n + 1$ is the u-invariant of some field.*

Proposition 3.8. *We have the inequality $u(k) \geqslant s(k)$*

To prove the Proposition, consider the quadratic form $\langle 1, \ldots, 1\rangle$ of dimension $s(k) - 1$. It doesn't represent -1, so $\langle 1, \ldots, 1\rangle$ of dimension $s(k)$ is anisotropic. Indeed, assume there exist $x_1, \ldots, x_{s(k)} \in k$, not all 0, such that $x_1 + \ldots + x_{s(k)} = 0$. We can consider that $x_{s(k)} \neq 0$. Then $-1 = \frac{x_1}{x_{s(k)}} + \ldots + \frac{x_{s(k)-1}}{x_{s(k)}}$, which is false.

Theorem 3.9. *If $u(k) < 2^n$, then $I^n(k) = 0$*

Theorem 3.10. *If k is not formally real, then $u(k) \leqslant \mathrm{Card}(k^\times / {k^\times}^2)$.*

Unlike for the case of the level, we don't have $u(K) \leqslant u(k)$ for all field extensions K/k. Instead :

Theorem 3.11 (Leep)**.** *For all extension K/k with finite degree $[K : k] = n$,*

$$u(K) \leqslant \frac{n+1}{2} u(k).$$