Brief overview of the theory of linear algebraic groups

Let k be a field.

1. Group schemes

A *group k-scheme* is a k-scheme G together with morphisms

$$m : G \times_k G \to G,\ i : G \to G,\ e : \mathrm{Spec}(k) \to G,$$

such that the following diagrams commute :

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd} \qquad \text{(associativity axiom),}$$

$$\begin{tikzcd}
G \arrow[rr, "(\mathrm{Id}_G, i)"] \arrow[dr] \arrow[dd, "(i, \mathrm{Id}_G)"'] & & G \times G \arrow[dd, "m"] \\
& \mathrm{Spec}(k) \arrow[dr, "i"] & \\
G \times G \arrow[rr, "m"'] & & G
\end{tikzcd} \qquad \text{(inverse axiom),}$$

$$\begin{tikzcd}
G \arrow[r, "(\mathrm{Id}_G, 1)"] \arrow[d, "(1, \mathrm{Id}_G)"'] \arrow[dr, "\mathrm{Id}_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd} \qquad \text{(unit axiom).}$$

An *affine* group k-scheme is a group k-scheme which is an affine scheme.
To a k-scheme X one associates its functor of points

$$\mathrm{F}_X : \left\{ \begin{array}{ccc} Alg_k & \to & Sets \\ R & \rightsquigarrow & X(R) = \hom_{k-\mathrm{sch}}(\mathrm{Spec}(R), X) \end{array} \right. .$$

For a group k-scheme G, all the sets $G(R)$ are naturally endowed with an abstract group structure functorial in k-algebras R, so actually $\mathrm{F}_G : Alg_k \to Grp$. Conversely, any factorization of the functor F_X through Grp for a given k-scheme X naturally endows X with a structure of a group k-scheme thanks to Yoneda's lemma.

An affine group k-scheme is then a representable functor $Alg_k \to Grp$.

If $G = \mathrm{Spec}(A)$ is an affine group k-scheme where A is an algebra, one have a correspondence :

$$\begin{cases} m : G \times_k G \to G & \longleftrightarrow \quad \Delta : A \to A \otimes_k A, \\ i : G \to G & \longleftrightarrow \quad S : A \to A, \\ e : \mathrm{Spec}(k) \to G & \longleftrightarrow \quad \epsilon : A \to k \end{cases}$$

with k-algebra homomorphism Δ, S, ϵ. The group axioms for m, i, and e translate into the commutative diagram :

$$\begin{tikzcd} A \otimes_k A \otimes_k A & A \otimes_k A \arrow[l, "\Delta\otimes\mathrm{Id}_A"'] \\ A \otimes_k A \arrow[u, "\mathrm{Id}_A\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \text{(co-associativity)},$$

$$\begin{tikzcd} A & & A \otimes_k A \arrow[ll, "\mathrm{multo}(\mathrm{Id}_A\otimes S)"'] \\ & k \arrow[ul] & \\ A \otimes_k A \arrow[uu, "\mathrm{multo}(S\otimes\mathrm{Id}_A)"] & & A \arrow[ll, "\Delta"'] \arrow[uu, "\Delta"'] \arrow[ul, "\epsilon"'] \end{tikzcd} \qquad \text{(co-inverse)},$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{Id}"] & A \otimes_k A \arrow[d, "\mathrm{multo}(\epsilon\otimes\mathrm{Id}_A)"] \\ A \otimes_k A \arrow[r, "\mathrm{multo}(\mathrm{Id}\otimes\epsilon)"'] & A \end{tikzcd} \qquad \text{(counit)}.$$

An algebra together with such homomorphisms Δ, S, ϵ is called an *Hopf algebra*.

An affine group k-scheme **is** an Hopf algebra.

An affine group k-scheme G is said to be *commutative* if for all k-algebras R, the abstract group $G(R)$ is commutative. This is equivalent to the commutativity of the diagram

$$\begin{tikzcd} G \times_k G \arrow[rr, "\mathrm{swap}"] \arrow[dr, "m"'] & & G \times_k G \arrow[dl, "m"] \\ & G & \end{tikzcd} .$$

Example 1.1. Here are introduced the basic examples of affine group schemes.

(i) The additive group $\mathbb{G}_a = \mathbb{G}_{a,k}$. $\mathbb{G}_a = \mathrm{Spec}(k[T])$ with Δ, S, ϵ characterized by the relations

$$\Delta(T) = T \otimes 1 + 1 \otimes T,\ S(T) = -T,\ \epsilon(T) = 0.$$

For every k-algebra R, $\mathbb{G}_a(R)$ is the additive group $(R, +)$.

(ii) The multiplicative group $\mathbb{G}_m = \mathbb{G}_{m,k}$. $\mathbb{G}_m = \mathrm{Spec}(k[T, T^{-1}])$ with Δ, S, ϵ characterized by the relations

$$\Delta(T) = T \otimes T,\ S(T) = T^{-1},\ \epsilon(T) = 1.$$

For every k-algebra R, $\mathbb{G}_a(R)$ is the multiplicative group $(R^\times, \times)$.

(iii) The general linear group $\mathrm{GL}_n = \mathrm{GL}_{n,k}$. It corresponds to the representable functor

$$R \rightsquigarrow \mathrm{GL}_n(R) \text{ the group of } n \times n \text{ invertible matrices.}$$

There is also the special linear group $\mathrm{SL}_n = \mathrm{SL}_{n,k}$. It corresponds to the representable functor

$$R \rightsquigarrow \mathrm{SL}_n(R) = \{g \in \mathrm{GL}_n(R) \mid \det(g) = 1\}.$$

(iv) The infinitesimal additive groups $\alpha_{p^r} = \alpha_{p^r,k}$.

Denote the characteristic of k by p and assume $p > 0$. The affine group scheme α_{p^r} corresponds to the representable functor

$$R \rightsquigarrow \{x \in R \mid x^{p^r} = 0\} \text{ as a subgroup of } (R, +).$$

We have $\alpha_{p^r} = \mathrm{Spec}(k[T]/(T^{p^r})$ with Δ, S, ϵ characterized by the relations

$$\Delta(\overline{T}) = \overline{T} \otimes 1 + 1 \otimes \overline{T},\ S(\overline{T}) = -\overline{T},\ \epsilon(\overline{T}) = 0,$$

where $\overline{T}$ is the class of T in $k[T]/(T^{p^r})$.

(v) The groups of roots of unity $\mu_n = \mu_{n,k}$.

The affine group scheme μ_n corresponds to the representable functor

$$R \rightsquigarrow \{x \in R^\times \mid x^n = 1\} \text{ as a subgroup of } (R^\times, \times).$$

We have $\mu_n = \mathrm{Spec}(k[T, T^{-1}]/(T^n - 1)$ with Δ, S, ϵ characterized by the relations

$$\Delta(\overline{T}) = \overline{T} \otimes \overline{T},\ S(\overline{T}) = \overline{T}^n,\ \epsilon(\overline{T}) = 1,$$

where $\overline{T}$ is the class of T in $k[T]/(T^n - 1)$.

2. Subgroup schemes and homomorphisms

2.1. **Subgroup schemes.** A *(closed) subgroup scheme* of an affine group scheme G is a closed subscheme H of G such that the morphisms m, i, e factor through H. We write $H \leqslant G$. Note that H is then affine.

A subgroup scheme $H \leqslant G$ yields natural inclusions $H(R) \subset G(R)$ between the sets of R-points and realize $H(R)$ as abstract subgroups of $G(R)$ for all k-algebras R.

Conversely, if H is a closed subscheme of the affine group scheme G such that $H(R)$ is a subgroup of $G(R)$ for all k-algebras R, then H is a subgroup scheme of G.

Example 2.1. We have $\alpha_{p^r} \leqslant \mathbb{G}_a$ ($p = \mathrm{char}(k) > 0$), $\mu_n \leqslant \mathbb{G}_m$, and $\mathrm{SL}_n \leqslant \mathrm{GL}_n$.

A subgroup scheme H of an affine group scheme G is *normal* if for all k-algebras R, the subgroups $H(R)$ are normal in $G(R)$ as abstract groups.

2.2. **Group homomorphisms.** A *group (scheme) homomorphism* $\varphi : G \to H$ between affine group schemes is a scheme morphism such that

$$m_H \circ (\varphi \times \varphi) = \varphi \circ m_G$$

where m_G and m_H denote the multiplication on G and H respectively. It is equivalent to the fact that $\varphi(R) : G(R) \to H(R)$ are abstract group homomorphisms for all k-algebras R.

Writing $G = \mathrm{Spec}(A)$ and $H\mathrm{Spec}(B)$, a group homomorphism $\varphi : G \to H$ corresponds to an Hopf algebra homomorphism $f : B \to A$, *i.e.* an algebra homomorphism such that

$$\begin{aligned} &\Delta_A \circ f = (f \otimes f) \circ \Delta_B, \\ &S_A \circ f = f \circ S_B, \\ &\epsilon_A \circ f = \epsilon_B, \end{aligned}$$

where Δ_C, S_C, and ϵ_C are the corresponding Hopf operations on $C = A,\, B$.

Given a group homomorphism $\varphi : G \to H$, the *kernel* of φ is the subgroup scheme $G \times_{\varphi, H, e_H} \mathrm{Spec}(k)$ where e_H is the unit morphism of H. For every k-algebra R, we have $(\ker\varphi)_R = \ker(\varphi_R)$ as subgroups of $G(R)$ where φ_R is the induced homomorphism $G(R) \to H(R)$.

Example 2.2. Here are basics kernels.

(i) Assume $p = \mathrm{char}(k) > 0$. Consider $\varphi_1 : \mathbb{G}_a \to \mathbb{G}_a$ given on R-points by $x \mapsto x^{p^r}$. Then $\ker\varphi_1 = \alpha_{p^r}$.

(ii) Consider $\varphi_2 : \mathbb{G}_m \to \mathbb{G}_m$ given on R-points by $x \mapsto x^n$. Then $\ker\varphi_2 = \mu_n$.

(iii) Consider $\det : \mathrm{GL}_n \to \mathbb{G}_m$ given on R-points by $g \mapsto \det(g)$. Then $\ker\det = \mathrm{SL}_n$.

Proposition 2.3. *Let $\varphi : G \to H$ be a group homomorphism between affine group k-schemes. Then φ is a closed embedding of schemes (making H a closed subgroup of G) if, an only if,* $\ker(\varphi) = 1$.

A *character* of the group scheme G is a group homomorphism $G \to \mathbb{G}_m$. The set of characters of G $\hom_{k-\mathrm{grp}}(G, \mathbb{G}_m)$ is an abstract group.

3. Change of base field

Let K/k be a field extension.

3.1. **Extension of scalars.** Let G an affine group k-scheme. Extending scalars from k to K yields an affine group scheme $G_K = G \times_k \mathrm{Spec}(K)$ over K. Writing $G = \mathrm{Spec}(A)$, we have $G_K = \mathrm{Spec}(A \otimes_k K)$. In terms of functor of points, considering $G : Alg_k \to Grp$, then G_K corresponds to the functor $Alg_K \to Grp$ which associates to any K-algebra R the group $G(R)$ where R is seen as a k-algebra.

3.2. **Weil restrictions.** Given a K-scheme X', consider the functor

$$\mathcal{R}_{K/k}(X') : \left\{ \begin{array}{ccc} Alg_k & \to & Sets \\ R & \rightsquigarrow & X'(R \otimes_k K) \end{array} \right. .$$

Proposition 3.1. *If X' is affine, then $\mathcal{R}_{K/k}(X')$ is representable by an affine k-scheme denoted by $R_{K/k}(X')$. If moreover X' is an affine group K-scheme, then $R_{K/k}(X')$ is an affine group k-scheme.*

If H' is a subgroup scheme of G', then $R_{K/k}(H')$ is a subgroup scheme of $R_{K/k}(G')$.

4. Scheme properties of affine group schemes

4.1. **Algebraic group schemes.** A group k-scheme is *algebraic (resp. locally algebraic)* if it is as a k-scheme.

Theorem 4.1. *If G is an algebraic affine group k-scheme, then G can be realized as a subgroup scheme of some general linear group scheme* GL_n.

A subgroup scheme of GL_n, which is necessarily algebraic, is said to be *linear*. Thanks to the Theorem, an algebraic affine group k-scheme is the same as a linear algebraic group k-scheme.

Note that a linear algebraic group k-scheme has finite dimension.

Example 4.2. **(i)** $\dim \alpha_{p^r} = 0$ where $p = \mathrm{char}(k) > 0$, and $\dim \mu_n = 0$.
(ii) $\dim \mathbb{G}_m = \dim \mathbb{G}_a = 1$.
(iii) $\dim \mathrm{GL}_n = n^2$ and $\dim \mathrm{SL}_n = n^2 - 1$.

4.2. **Smoothness.** A group k-scheme is *smooth* if it is as a k-scheme.

Proposition 4.3. *Let G be an affine group k-scheme. Then G is smooth if, and only if, G is geometrically reduced (i.e. $G_{\bar{k}}$ is reduced). Writing $G =$* $\mathrm{Spec}(A)$ *, this means that the $\bar{k}$-algebra $A \otimes_k \bar{k}$ is reduced.*

Example 4.4. **(i)** $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n, and SL_n are all smooth.
(iii) When n is not divisible by $\mathrm{char}(k)$, μ_n is smooth.
(iii) When $p = \mathrm{char}(k) > 0$, α_{p^r} and μ_{p^r} are not smooth.

4.3. **Connectedness.**

Proposition 4.5. *Let G be a linear algebraic group k-scheme. If G is connected, then it is geometrically irreducible.*

In general, for a linear algebraic group k-scheme G, it is Noetherian, hence G decomposes in connected components $G = X_1 \coprod \cdots \coprod X_n$. Let G° be the connected component which contains the image of $e : \mathrm{Spec}(k) \to G$. Then G° is a closed and open subscheme of G; it is a normal subgroup scheme. For any field extension K/k, the canonical homomorphism $(G^\circ)_K \to (G_K)^\circ$ is an isomorphism.

5. Quotients of group schemes

Let G be a linear algebraic group k-scheme and take $N \leqslant G$ to be a normal subgroup. A *quotient* of G by N is a linear algebraic group $k$scheme H together with a group homomorphism $q : G \to H$ such that q is faithfully flat and $\ker(q) = N$.

Theorem 5.1. *Given a linear algebraic group scheme G and a normal subgroup scheme $N \leqslant G$, a quotient of G by N always exists, and is "unique".*

Because of the unicity property, we talk about "the" quotient.

The quotient of G by N is denoted by $q : G \to G/N$.

Remark 5.2. The quotient map $G \to G/N$ is surjective but for any k-algebra R, the homomorphism $G(R) \to G/N(R)$ is not necessarily surjective.

Proposition 5.3. *Let G, H be linear algebraic groups and let $\varphi : G \to H$ be a group homomorphism. If H is reduced, then φ is faithfully flat if, and only if, φ is surjective.*

Proposition 5.4. *If G is smooth (resp. connected), then every quotient of G is smooth (resp. connected).*

A sequence $G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3$ of linear algebraic group schemes is said to be *exact* if $\varphi_2 \circ \varphi_1 = 1$ and the induced homomorphism $\varphi_1 : G_1 \to \ker \varphi_2$ is faithfully flat.

Example 5.5. The sequence $1 \to G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \to 1$ is exact if, and only if, φ_1 is a closed embedding, turning G_1 into a subgroup scheme of G_2, and G_3 is the quotient G_2/G_1.

6. Tori

A *k-torus* is a group k-scheme T such that $T_{\bar{k}}$ is isomorphic to a product $\mathbb{G}_m^r = \mathbb{G}_m \times_k \cdots \times_k \mathbb{G}_m$ over $\bar{k}$ ($\mathbb{G}_m^r$ corresponds to the functor of points $\mathbb{G}_m^r(R) = (R^\times)^r$). A *split* k-torus is a group k-scheme T such that T is isomorphic to a product $\mathbb{G}_m^r$ over the base field k.

Remark 6.1. Tori are smooth and connected.

Example 6.2.

(i) Consider the group $\mathbb{R}$-scheme T whose R-points are

$$T(R) = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{GL}_2(R) \mid a^2 + b^2 = 1\}.$$

Then $T_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}}$, so T is an $\mathbb{R}$-torus.

(ii) Let L/k be a finite separable field extension, and let $T = \mathrm{R}_{L/k}(\mathbb{G}_{m,L})$. Then T is a k-torus with $T_L \simeq \mathbb{G}_{m,L}^{[L:k]}$.

Proposition 6.3. *Every k-torus becomes split over the separable closure k_s of k. This implies that for any torus T, there is a finite separable extension L/k such that $T_L \simeq \mathbb{G}_{m,L}^r$ for some integer r.*

Consequence 6.4. *Let k'/k be a finite purely inseparable extension and let T be a k-torus. If $T_{k'}$ is split, then T is already split over k.*

The "opposite" of a split torus is an *anisotropic* torus : this is a k-torus T such that there is no non-trivial character $T \to \mathbb{G}_m$.

Given a smooth algebraic group k-scheme G, a *maximal torus* of G is a subtorus $T \leqslant G$ which is not strictly contained in another torus.

Maximal tori exist for dimension reasons.

Theorem 6.5. *Let G be a smooth linear algebraic group k-scheme and $T \leqslant G$ be a maximal torus.*

- *For all field extension K/k, $T_K \leqslant G_K$ is still a maximal torus.*
- *Over a separably closed field $k = k_s$, all maximal tori of G are conjugated by elements of $G(k)$. Thus maximal tori have the same dimension over an arbitrary field k.*

Example 6.6. In GL_n, the subgroup scheme T whose R-points are given by the diagonal matrices in $\mathrm{GL}_n(R)$ for all k-algebras R, is a maximal torus.