On (Brauer) cohomological invariants of linear algebraic groups

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Online seminar
Quafratic forms, linear algebraic groups and beyond

September 28, 2020

Sommaire

Let k be any field and G a smooth algebraic k-group.

For all field extension K/k :
G-torsor over K : K-scheme X equipped with a (right) G action such that

- the action is faithful and transitive, $\begin{array}{ccc} X \times G & \to & X \times_K X \\ (x, g) & \mapsto & (x, x \cdot x) \end{array}$;
- $X(K_s) \neq \emptyset$ for K_s a separable closure of K.

$\mathrm{H}^1(K, G)$ first Galois cohomology set. It is the set of isomorphism classes of G-torsors over K. Pointed by the classe of G_K with right multiplication.

Example

1. O_n the orthogonal group. The set $\mathrm{H}^1(K, O_n)$ classifies the isometric classes of quadratic forms of rank n over K^n.
2. PGL_n the projective linear group. The set $\mathrm{H}^1(K, \mathrm{PGL}_n)$ classifies central simple algebras of degree n over K.

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$\longrightarrow$ Functor $\mathrm{H}^1(\star, G) : \underline{Fields}_k \to \underline{Sets}$
where

- $\underline{Fields}$ the category of field extensions of k,
- $\underline{Sets}$ the category of sets.

Why study $\mathrm{H}^1(k, G) \longrightarrow$ get 1) information on the geometry of G and 2) information on A's forms to understand the obstruction of Galois descent problems (*e.g.* quadratic forms).
Idea $\longrightarrow$ in order to study $\mathrm{H}^1(k, G)$, one can benefit from the cohomological aspect of H^1 and see how it behaves when mapped into Galois cohomology group : H^1 **not a group in general**.
Let $d \geqslant 0$ be an integer and M a Galois module for k.

$\longrightarrow \mathrm{H}^d(\star, M) : \underline{\mathit{Fields}}_k \to \underline{\mathit{Ab}}$ where $\underline{\mathit{Ab}}$ is the category of abelian groups

Definition (Serre)

A *degree d cohomological invariant of G with coefficients in M* is a natural transformation from the functor $\mathrm{H}^1(\star, G)$ into the functor $\mathrm{H}^d(\star, M)$.

A degree d cohomological invariant of G with coefficients in M is a collection of set maps

$$i_K : \mathrm{H}^1(K, G) \to \mathrm{H}^d(K, M)$$

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One has a decomposition of abelian groups

$$\mathrm{Inv}^d(G, M) = \underbrace{\mathrm{H}^d(k, M)}_{\text{constant inv.}} \oplus \underbrace{\mathrm{Inv}^d(G, M)_0}_{\textit{normalized}\text{ inv.}}.$$

Important examples (d integer) :

- $M = \mathbb{Z}/2$ lorsque $\mathrm{char}(k) \neq 2$;
- $M = \mu_n^{\otimes(d-1)}$ for n coprime to $\mathrm{char}(k)$ (μ_n group of n-th roots of unity) ;
- more generally $\mathbb{Q}/\mathbb{Z}(d-1)$: sum of

$$\varinjlim_m \mu_{q^m}^{\otimes(d-1)} \text{ pour } q \neq \mathrm{char}(k) \text{ premier}$$

$$\text{«} \varinjlim_m \mu_{p^m}^{\otimes(d-1)} \text{ » pour } p = \mathrm{char}(k) > 0$$

$\mathbb{Q}/\mathbb{Z}(d-1)$ is a Galois module also define as an étale sheaf for every k-scheme.
One has $\mathrm{H}^2(L, \mathbb{Q}/\mathbb{Z}(1)) \simeq \mathrm{Br}(L)$ for any field L.

$$\longrightarrow \mathrm{Inv}^2(*, \mathbb{Q}/\mathbb{Z}(1))_0 =: \mathrm{Inv}(*, \mathrm{Br})_0.$$

For $M = \mathbb{Z}/2$ when $\mathrm{char}(k) \neq 2$: the Stiefel-Whitney invariants

For quadratic forms,

Theorem (Serre)

L'anneau (pour le cup-produit)

$$\bigoplus_{d\geqslant 0}\mathrm{Inv}^d(\mathrm{O}_n,\mathbb{Z}/2\mathbb{Z}))$$

du groupe orthogonal est un $\mathrm{H}^*(k,\mathbb{Z}/2\mathbb{Z})$*-module libre engendré par les classes de Stiefel-Whitney.*

In degree 1,

Proposition

For every torsion Galois module M, if G is connected then

$$\mathrm{Inv}^1(G,M)_0=0.$$

In degree 2,

Proposition (Blinstein-Merkurjev)

In degree 3,

Theorem (Rost)

For G absolutly simple, simply connected, $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_0$ is a finite cyclic group.

Wide generalization :

- For every semi-simple groups : Merkurjev,
- For tori : Blinstein-Merkurjev,
- For every split reductive groups : Lackman-Merkurjev.

Sommaire

-Let k be a field.
(No restriction on k : it is finite or infinite, of zero or positive characteristic, perfect or non-perfect.)
-Let G be a smooth and connected linear algebraic k-group.

Denote by $\mathrm{Ext}^1(G, \mathbb{G}_m)$ the group of equivalence classes of extensions

$$1 \to \mathbb{G}_m \to H \to G \to 1,$$

equipped with the Baer sum.

Theorem (L.)

There is an isomorphism, natural w.r.t. G,

$$\mathrm{Ext}^1(G, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{Inv}(G, \mathrm{Br})_0 \ \left(= \mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_0\right).$$

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Recall : one has a canonical homomorphism $\mathrm{Ext}^1(G,\mathbb{G}_m) \hookrightarrow \mathrm{Pic}(G)$.

• Si G *réductif* ou k *parfait*, on a une identification (Sansuc, Colliot-Thélène)

$$\mathrm{Ext}^1(G,\mathbb{G}_m) \simeq \mathrm{Pic}(G).$$

$\longrightarrow$ One recovers Blinstein-Merkurjev's result.
In that case $\mathrm{Pic}(G)$ is finite.

• More generally, if G is *unirational* over k_s, then $\mathrm{Ext}^1(G,\mathbb{G}_m) \simeq \mathrm{Pic}(G)$ and $\mathrm{Pic}(G)$ is finite (Gabber-Rosengarten).

- In general, $\mathrm{Ext}^1(G, \mathbb{G}_m) \neq \mathrm{Pic}(G)$ and $\mathrm{Ext}^1(G, \mathbb{G}_m)$ can be infinite.

For instance, it is the case for some unipotent groups over an imperfect field (Achet, Rosengarten, Totaro).
Then $\mathrm{Ext}^1(G, \mathbb{G}_m)$ is a p-primary torsion group, $p = \mathrm{car}(k)$.

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Define

$$\Phi_G : \left\{ \begin{array}{ccc} \mathrm{Ext}^1(G,\mathbb{G}_m) & \to & \mathrm{Inv}(G,\mathrm{Br})_0 \\ E & \mapsto & \delta_E^{\cdot} \end{array} \right. .$$

Proposition (L.)

For every smooth and connected linear algebraic k-group, Φ_G is a group isomorphism.

Recall : Weil restriction.

K/k field extension and X a K-scheme.

$\mathrm{R}_{K/k}(X)$ is the k-scheme characterized by the property

$$\mathrm{R}_{K/k}(X)(A) = X(K \otimes_k A)$$

for every k-algebra A.

Call p the characteristic exponent of k.
-Let k'/k finite, purely inseparable and h the least integer for which $(k')p^h \subseteq k$.

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-Let k'/k finite, purely inseparable and h the least integer for which $(k')p^h \subseteq k$.
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Write $\widehat{\mu'}$ for the character group of μ' defined overr k'.

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Proposition (L.)

For $G = \mathrm{R}_{k'/k}(G'/\mu')$ and $G = \mathrm{R}_{k'/k}(G')/\mathrm{R}_{k'/k}(\mu')$, one has

$$\mathrm{Inv}(G, \mathrm{Br})_0 \simeq p^h \widehat{\mu'}.$$

Example

Let p be a prime and k'/k finite, purely inseparable of characteristic p. Define h as above. Then for every integer $n \geqslant 1$, writing $G = \mathrm{R}_{k'/k}(\mathrm{SL}_{p^n,k'})/\mathrm{R}_{k'/k}(\mu_{p^n,k'})$, one has

$$\mathrm{Inv}(G, \mathrm{Br})_0 \simeq p^h \cdot \mathbb{Z}/p^n\mathbb{Z}.$$

Remark 1 : there is only p-primary torsion.

Remark 2 : when $h \geqslant n$, there is no non-trivial normalized invariant.

Sommaire

Soit G un groupe algébrique affine, lisse sur un corps k.
(Serre/Merkurjev) On se donne une immersion

$$G \hookrightarrow \mathrm{GL}_n.$$

Le morphisme $\xi : \underbrace{\mathrm{GL}_n}_{=:U} \to \underbrace{\mathrm{GL}_n/G}_{=:X}$ est un G-torseur.

It is a scheme. This is a *versal* torseur : for any K/k with K infinite and any G-torsor Y/K, there exists a morphism $f : \mathrm{Spec}(K) \to X$ such that Y/K is the pull back of ξ through f,

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & U \arrow[d, "\xi"] \\ \mathrm{Spec}(K) \arrow[r, "f"'] & X \end{tikzcd}$$

Following Serre/Merkurjev one defines

$$\mathrm{Br}(X)_{\text{éq}} \to \mathrm{Inv}(G, \mathrm{Br}),$$
$$\mathrm{H}^0_{\text{Zar}}(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1)))_{\text{éq}} \to \mathrm{Inv}(G, \mathrm{Br}).$$

where

- $()_{\text{éq}}$: sous-groupe des éléments *équilibrés* ;
- $\mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1))$ faisceautisé pour Zariski de $V \to \mathrm{H}^2(V, \mathbb{Q}/\mathbb{Z}(1)$ sur X.

Modèle simplicial de torseur classifiant (Deligne) :
Let

$$\mathrm{E}G^\bullet = (G^{n+1})_{n\in\mathbb{N}}$$

and

$$\mathrm{B}G^\bullet := \mathrm{E}G^\bullet/G = (G^{n+1}/G)_{n\in\mathbb{N}}$$

for the right diagonal action of G on G^n.

To any G torsor $Y \to Z$, one associates a simplicial scheme $[Y|G]^\bullet$ such that

- there is a morphism $[Y|G]^\bullet \to \mathrm{B}G^\bullet$,
- $\mathrm{H}^2([Y|G]^\bullet, \mathbb{G}_m) \simeq \mathrm{H}^2(Z, \mathbb{G}_m)$.

Define $\mathrm{H}^2(\mathrm{B}G^\bullet, \mathbb{G}_m) \to \mathrm{Inv}(G, \mathrm{Br})$ as follows : Let $\alpha \in \mathrm{H}^2(\mathrm{B}G^\bullet, \mathbb{G}_m)$; if Y/K is a G-torsor over an extension K/k, there is a map $i_\alpha(Y) : \mathrm{H}^2(\mathrm{B}G^\bullet, \mathbb{G}_m) \to \mathrm{H}^2(K, \mathbb{G}_m) = \mathrm{Br}(K)$. One checks that $i_\alpha(Y)$ doesn't depend on the isomorphism class of Y/K.

Diagramme :

$$\begin{tikzcd}
\mathrm{H}^2(\mathrm{B}G^\bullet, \mathbb{G}_m)/\mathrm{H}^2(k, \mathbb{G}_m) \arrow[r] & \mathrm{Br}(X)_{\text{éq}}/\mathrm{Br}(k) \arrow[r] & \mathrm{Inv}(G, \mathrm{Br})_0 \\
& \mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(1)))_{\text{éq}}/\mathrm{H}^2(k, \mathbb{G}_m) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

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\end{tikzcd}$$

- (3) isomorphisme par conjecture de Gersten
- (4) isomorphisme par Blinstein-Merkurjev

On se retrouve dans la situation :

$$\mathrm{H}^2(\mathrm{B}G^\bullet, \mathbb{G}_m)/\mathrm{H}^2(k, \mathbb{G}_m) \underset{(1)}{\longrightarrow} \mathrm{Br}(X)_{\text{éq}}/\mathrm{Br}(k) \underset{(2)}{\overset{\sim}{\to}} \mathrm{Inv}(G, \mathrm{Br})_0.$$

En écrivant

$$\mathrm{Br}(X) = \mathrm{H}^2([U|G]^\bullet, \mathbb{G}_m) \text{ (où } U = \mathrm{GL}_n)$$

et des calculs de cohomologie sur des espaces simpliciaux, on trouve :

$$\mathrm{Ext}^1(G, \mathbb{G}_m) \overset{\sim}{\to} \mathrm{H}^2(\mathrm{B}G^\bullet, \mathbb{G}_m)/\mathrm{H}^2(k, \mathbb{G}_m) \overset{\sim}{\to} \mathrm{Br}(X)_{\text{éq}}/\mathrm{Br}(k).$$

Merci à tous de votre attention !